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A Kuratowski–Mrówka theorem in approach theory

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Abstract

In this paper we give a number of arguments why, in approach theory, the notion of compactness which from the intrinsic categorical point of view seems most satisfying is 0-*compactness*, i.e., measure of compactness equal to zero. It was already known from [R. Lowen, Kuratowski's measure of noncompactness revisited, Quart. J. Math. Oxford 39 (1988) 235–254] that measure of compactness has good properties and good interpretations for both topological and metric approach spaces. Here, introducing notions of closed and proper mappings in approach theory, which satisfy all the intrinsic categorical axioms put forth in [Clementino et al., A functional approach to topology, in: M.C. Pedicchio, W. Tholen (Eds.) Categorical Foundations Special Topics in Order, Topology, Algebra, and Sheaf Theory, Cambridge University Press, 2003], we prove fundamental results concerning these concepts, also linked to 0-compactness, and we give a Kuratowski–Mrówka-type characterization of 0-compactness.

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1. Preliminaries

For all categorical notions we refer to [1,4]. Given a set X, $\mathbf{U}(X)$ stands for the set of all ultrafilters on X. Unless otherwise mentioned all spaces in this paper are assumed to be approach spaces and we refer to [6–9] for the basic theory. For such a space the

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generic notations for its distance, limit operator and gauge are, respectively, δ, λ and $\mathcal{G}$. If necessary we may give these symbols sub- or superscripts, but in general it will be clear from the context on which space they are defined. In [9] all necessary transition formulas for going from one structure to another can be found. However, whenever we need a particular formula, we will recall it shortly prior to using it. The *measure of compactness* (of X) is defined as

$$\mu_c(X) := \sup_{\mathcal{U}\in\mathbf{U}(X)} \inf_{x\in X} \lambda\mathcal{U}(x) = \sup_{\varphi\in\mathcal{G}^X} \inf_{\substack{Y\subset X\\ Y \text{ finite}}} \sup_{z\in X} \inf_{x\in Y} \varphi(x)(x,z).$$

For metric spaces this notion originally goes back to Kuratowski [5] and it was extensively studied in the setting of approach spaces in [6].

We will call an approach space X 0-*compact* if $\mu_c(X) = 0$. Note that this is not the same as saying that the topological coreflection of X [9] is compact. This latter property is, in general, stronger and is referred to as being *compact*, this in keeping with our convention to say that an approach space has a topological property if and only if the topological coreflection (or underlying topological space) has that property.

Using respectively the first and second expression for μ_c the following is easily seen to hold.

Proposition 1.1 [6]. *A topological approach space is* 0*-compact if and only if it is compact, and a pseudometric approach space is* 0*-compact if and only if it is totally bounded.*

It was also shown in [6] that measure of compactness itself satisfies a general form of the Tychonoff theorem.

Theorem 1.2 (Tychonoff) [6]. *For a family of approach spaces $(X_j)_{j\in J}$ the following formula holds*:

$$\mu_c\left(\prod_{j\in J} X_j\right) = \sup_{j\in J} \mu_c(X_j).$$

As a consequence of Theorem 1.2 a Tychonoff theorem of course also holds for 0-compactness.

Corollary 1.3 (Tychonoff). *The product of a family of approach spaces is* 0*-compact if and only if each member of the family is* 0*-compact.*

The following result was also shown in [6].

Proposition 1.4 [6]. *If $f : X \to Y$ is a surjective contraction then $\mu_c(Y) \leqslant \mu_c(X)$.*

And this too has an immediate corollary for 0-compactness.

Corollary 1.5. *If $f : X \to Y$ is a surjective contraction and X is* 0*-compact then Y is* 0*-compact.*

We recall that if $A \subset X$ then θ_A denotes the indicator of A, i.e. the $[0,\infty]$-valued function which takes on the value 0 on A and ∞ outside of A. We also recall that given a function $f : X \to Y$, $\mu \in [0,\infty]^X$ and $\nu \in [0,\infty]^Y$ the image of μ is defined as $f(\mu)(y) := \inf_{x \in f^{-1}(y)} \mu(x)$ and the preimage of ν is defined as $f^{-1}(\nu) := \nu \circ f$. The functions $f^{-1}(\nu)$ and $f(\mu)$ define a pair of adjoint mappings and so one is completely determined by the other via the relation which says that for all $\mu \in [0,\infty]^X$ and $\nu \in [0,\infty]^Y$: $\nu \leqslant f(\mu) \Longleftrightarrow f^{-1}(\nu) \leqslant \mu$.

2. Closed and proper contractions

Approach spaces form a topological construct [9], denoted **Ap**. The morphisms are so-called contractions. Given approach spaces X and Y, a function $f : X \to Y$ is called a contraction if for all $x \in X$ and $A \subset X$ we have $\delta(f(x), f(A)) \leqslant \delta(x, A)$ or shortly, if for all $A \subset X$ we have $\delta_{f(A)} \circ f \leqslant \delta_A$. It is therefore not surprising that a concept of closed map in **Ap** should involve a form of expansiveness, as opposed to contractiveness. The same is true for a notion of open maps, but we will not be concerned with this in the present paper. Note that the characterization of contractions above can also be written as $f^{-1}(\delta_{f(A)}) \leqslant \delta_A$ or, equivalently $\delta_{f(A)} \leqslant f(\delta_A)$.

Definition 2.1. Given approach spaces X and Y, a function $f : X \to Y$ is called *closed* if for all $A \subset X$ we have $f(\delta_A) \leqslant \delta_{f(A)}$.

Proposition 2.2. *If X and Y are topological approach spaces then a map $f : X \to Y$ is closed if and only if it is closed in the topological sense.*

Proof. Immediate from the definition. □

Proposition 2.3. *Given approach spaces X and Y, and a function $f : X \to Y$ the following are equivalent*:

(1) *f is an injective closed contraction,*
(2) *f is an embedding such that $\delta_{f(X)} = \theta_{f(X)}$.*

Proof. If f is an injective closed contraction then it follows immediately from the definitions that for all $x \in X$ and $A \subset X$

$$\delta\big(f(x), f(A)\big) = \delta(x, A),$$

and hence f is an embedding. Moreover, $\delta_{f(X)} = f(\delta_X) = f(\theta_X) = \theta_{f(X)}$. Conversely, if f is an embedding and $\delta_{f(X)} = \theta_{f(X)}$ then f is obviously injective and a contraction. Let $A \subset X$ and $y \in Y$. If $y \in f(X)$ and $x \in X$ is the unique point such that $f(x) = y$ then

$$f(\delta_A)(y) = \delta(x, A) = \delta\big(f(x), f(A)\big) = \delta_{f(A)}(y),$$

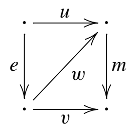
and if $y \notin f(X)$ then

$$f(\delta_A)(y) = \infty = \theta_{f(X)}(y) = \delta_{f(X)}(y) \leqslant \delta_{f(A)}(y).$$

Hence $f(\delta_A) \leqslant \delta_{f(A)}$ which shows that f is closed. □

For a category with a given factorization structure $(\mathcal{E}, \mathcal{M})$, satisfying properties (F0)–(F2) in [2], Clementino, Giuli and Tholen formulate three further axioms which a class $\mathcal{F}$ of morphisms has to fulfil for it to be a viable class of closed morphisms in that category. We first recall the axioms (F0)–(F2):

(F0) $\mathcal{M}$ is a class of monomorphisms and $\mathcal{E}$ is a class of epimorphisms and both are closed under composition with isomorphisms,
(F1) every morphism f decomposes as $f = m \circ e$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$,
(F2) every $e \in \mathcal{E}$ is orthogonal to every $m \in \mathcal{M}$, that is, given any morphisms u and v such that $m \circ u = v \circ e$ there exists a unique morphism w making the following diagram commutative



The axioms on $\mathcal{F}$ are:

(F3) $\mathcal{F}$ contains all isomorphisms and is closed under composition,
(F4) $\mathcal{F} \cap \mathcal{M}$ is stable under pullbacks,
(F5) whenever $g \circ f \in \mathcal{F}$ and $f \in \mathcal{E}$ then $g \in \mathcal{F}$.

Note that in the abstract categorical setting these axioms necessarily are formulated in terms of morphisms, whereas our notion of closedness also makes sense for an arbitrary function, hence the supplementary (and required) condition in Corollary 2.6.

In our case, we consider (one of) the usual factorization structure(s) $(\mathcal{E}, \mathcal{M})$ on **Ap**, where $\mathcal{E}$ are the epimorphisms (i.e., the surjective contractions) and $\mathcal{M}$ are the extremal monomorphisms (i.e., the embeddings). As in any topological construct, this factorization structure satisfies the aforementioned conditions (F0)–(F2). From now on,

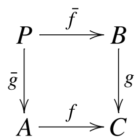
$$\mathcal{F} := \text{the class of all closed contractions.}$$

That isomorphisms are closed and that closed contractions are stable under composition is evident from the definition. This implies that $\mathcal{F}$ satisfies axiom (F3).

We will now point out that $\mathcal{F}$ also satisfies the remaining two axioms with regard to the given factorization structure.

Proposition 2.4. *$\mathcal{F} \cap \mathcal{M}$ is stable under pullbacks.*

Proof. Consider the pullback diagram



where f is a closed embedding. We mention that, as in all topological constructs, we can take $P = \{(a,b) \in A \times B \mid f(a) = g(b)\}$ where $A \times B$ carries the product structure and P the subspace structure and where $\bar{f}$ and $\bar{g}$ are the restrictions of the projections. Since $\mathcal{M}$ is stable under pullbacks we already obtain that $\bar{f}$ is an embedding. Hence it remains to show that $\delta_{\bar{f}(P)} = \theta_{\bar{f}(P)}$. Since g is a contraction we have that

$$\begin{aligned}
\theta_{\bar{f}(P)} &= \theta_{g^{-1}(f(A))} \\
&= g^{-1}(\theta_{f(A)}) \\
&= g^{-1}(\delta_{f(A)}) \\
&\leqslant g^{-1}(\delta_{gg^{-1}f(A)}) \\
&\leqslant \delta_{g^{-1}(f(A))} \\
&= \delta_{\bar{f}(P)}.
\end{aligned}$$

Since the other inequality always holds, again by Proposition 2.3, this proves that $\bar{f}$ is in $\mathcal{F} \cap \mathcal{M}$. □

This proves that axiom (F4) is fulfilled.

Proposition 2.5. *If $g \circ f$ is closed and f is a surjective contraction then g is closed.*

Proof. Let $f : X \to Y$, $g : Y \to Z$ be as stated and let $B \subset Y$. Then we have

$$\begin{aligned}
g(\delta_B) &= g \circ f\bigl(f^{-1}(\delta_B)\bigr) \\
&\leqslant g \circ f(\delta_{f^{-1}(B)}) \\
&\leqslant \delta_{g \circ f(f^{-1}(B))} \\
&= \delta_{g(B)},
\end{aligned}$$

which shows that g is closed. □

Corollary 2.6. *If g is a contraction, $g \circ f \in \mathcal{F}$ and $f \in \mathcal{E}$ then $g \in \mathcal{F}$.*

This proves, finally, that also axiom (F5) is fulfilled.

From [2] we adopt the following definition of proper morphism.

Definition 2.7. A contraction $f : X \to Y$ is called a *proper contraction* if it belongs stably to $\mathcal{F}$, i.e., whenever

$$\begin{array}{ccc}
W & \xrightarrow{\bar{f}} & Z \\
{\scriptstyle \bar{g}}\downarrow & & \downarrow{\scriptstyle g} \\
X & \xrightarrow{f} & Y
\end{array}$$

is a pullback diagram, $\bar{f} \in \mathcal{F}$.

In view of the following property, verifying properness of a contraction can be done by the simple criterion in 2.9, [2].

Proposition 2.8. *$\mathcal{F}$ is stable under restrictions.*

Proof. Let $f : X \to Y$ be a closed contraction, let $Z \subset Y$ and consider the restriction

$$g := f|_{f^{-1}(Z)} : f^{-1}(Z) \to Z.$$

Let $A \subset f^{-1}(Z)$ and $y \in Z$, then

$$g(\delta_A)(y) = \inf_{x \in g^{-1}(y)} \delta_A(x) = \inf_{x \in f^{-1}(y)} \delta_A(x) \leqslant \delta_{f(A)}(y) = \delta_{g(A)}(y),$$

which shows that g is closed. □

Proposition 2.9. *A contraction $f : X \to Y$ is proper if and only if for each approach space Z the map $f \times 1_Z : X \times Z \to Y \times Z$ is closed.*

From now on, the class of proper contractions will be denoted by $\mathcal{F}^*$. From the definition, it is immediately clear that $\mathcal{F}^* \subset \mathcal{F}$. The following results are also immediate consequences of the general results proved in [2].

Proposition 2.10. *The class of proper contractions fulfills the following stability properties*:

(1) *Proper contractions are stable under composition.*
(2) *Closed embeddings are proper contractions.*
(3) *$\mathcal{F}^*$ is the largest pullback-stable subclass of $\mathcal{F}$.*
(4) *If $g \circ f$ is a proper contraction and g is an injective contraction, then f is a proper contraction.*

Proposition 2.11. *If X and Y are topological approach spaces then a contraction $f : X \to Y$ is proper if and only if it is proper in the topological sense.*

Proof. Immediate from the fact that **Top** is concretely reflective in **Ap**. □

3. Proper contractions and 0-compactness

We consider the following ultrafilter spaces. Fix a set X and an ultrafilter $\mathcal{U}$ on X. Take $\omega \notin X$, let $X_{\mathcal{U}} := X \cup \{\omega\}$ and put $\mathcal{U}_\omega$ the ultrafilter on $X_{\mathcal{U}}$ generated by $\mathcal{U}$. The following defines a topology on $X_{\mathcal{U}}$; the only convergent ultrafilters are the point filters, which converge to their defining points and the filter $\mathcal{U}_\omega$ which converges to ω. The approach space generated by this topology has as limit operator (on ultrafilters)

$$\lambda^{\mathcal{U}}\mathcal{V}(x) := \begin{cases} 0 & (\mathcal{V} = \text{stack } x \text{ and } x \in X) \text{ or } (\mathcal{V} = \mathcal{U}_\omega \text{ and } x = \omega), \\ \infty & \text{all other cases,} \end{cases}$$

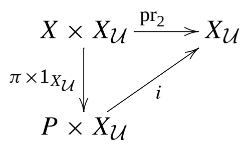
and as distance

$$\delta^{\mathcal{U}}(x, A) := \begin{cases} 0 & x \in A \text{ or } (A \in \mathcal{U}_\omega \text{ and } x = \omega), \\ \infty & \text{all other cases.} \end{cases}$$

This gives a special (albeit topological) case of the approach spaces also considered in [3]. We recall the following formulas which will be required in the sequel [9]. If $\mathcal{U}$ is an ultrafilter then $\lambda\mathcal{U} = \sup_{U \in \mathcal{U}} \delta_U$ and conversely if $A \subset X$ then $\delta_A = \inf\{\lambda\mathcal{U} \mid \mathcal{U} \in \mathbf{U}(X),\ A \in \mathcal{U}\}$. If $(f_j : X \to X_j)_{j \in J}$ is a source in **Ap** then the initial structure is characterized via its limit operator by the formula $\lambda\mathcal{G} = \sup_{j \in J} \lambda_j f_j(\mathcal{G}) \circ f_j$, for any filter $\mathcal{G}$ on X.

Proposition 3.1. *If for a one-point space P the unique morphism $\pi : X \to P$ is proper then X is* 0*-compact.*

Proof. Let $\mathcal{U}$ be an ultrafilter on X, let λ stand for the limit operator on X, and let $\lambda^{(\mathcal{U})}$ and $\delta^{(\mathcal{U})}$ stand for the limit operator and distance on the product space $X \times X_{\mathcal{U}}$, where $X_{\mathcal{U}}$ is the ultrafilter space defined above. Consider the diagram below, where i is the evident isomorphism:



Since π is a proper contraction, it follows that $\pi \times 1_{X_{\mathcal{U}}}$ is closed and hence so is pr_2. This implies that

$$\mathrm{pr}_2\big(\delta_\Delta^{(\mathcal{U})}\big)(\omega) \leqslant \delta^{\mathcal{U}}_{\mathrm{pr}_2(\Delta)}(\omega),$$

where $\Delta := \{(x, x) \in X \times X_{\mathcal{U}} \mid x \in X\}$. For the right-hand side of this inequality, since $X \in \mathcal{U}_\omega$, we clearly have

$$\delta^{\mathcal{U}}_{\mathrm{pr}_2(\Delta)}(\omega) = \delta^{\mathcal{U}}(\omega, X) = 0.$$

We now calculate the left-hand side.

$$\begin{aligned}
\mathrm{pr}_2\big(\delta_\Delta^{(\mathcal{U})}\big)(\omega) &= \inf_{(x,y) \in (\mathrm{pr}_2)^{-1}(\omega)} \delta_\Delta^{(\mathcal{U})}(x, y) \\
&= \inf_{x \in X} \delta_\Delta^{(\mathcal{U})}(x, \omega) \\
&= \inf_{x \in X} \inf_{\substack{\mathcal{V} \text{ ultra} \\ \Delta \in \mathcal{V}}} \lambda^{(\mathcal{U})}\mathcal{V}(x, \omega) \\
&= \inf_{x \in X} \inf_{\substack{\mathcal{V} \text{ ultra} \\ \Delta \in \mathcal{V}}} \lambda \mathrm{pr}_1\mathcal{V}(x) \vee \lambda^{\mathcal{U}} \mathrm{pr}_2\mathcal{V}(\omega).
\end{aligned}$$

Note that the only way $\lambda^{\mathcal{U}}\mathrm{pr}_2\mathcal{V}(\omega)$ can be different from ∞ (and then necessarily equal to 0) is if $\mathrm{pr}_2\mathcal{V} = \mathcal{U}_\omega$. This happens precisely when $\mathcal{V} = \mathcal{U}_\Delta$ where $\mathcal{U}_\Delta$ is the filter generated by $\{(U \times U) \cap \Delta \mid U \in \mathcal{U}\}$. Hence we find that

$$\begin{aligned}\mathrm{pr}_2\big(\delta_\Delta^{(\mathcal{U})}\big)(\omega) &= \inf_{x\in X} \lambda \mathrm{pr}_1 \mathcal{U}_\Delta(x)\\ &= \inf_{x\in X} \lambda\mathcal{U}(x).\end{aligned}$$

Thus $\inf_{x\in X} \lambda\mathcal{U}(x) = 0$, which by the arbitrariness of $\mathcal{U}$ shows that X is 0-compact. □

Lemma 3.2. *If $A \subset X$ is 0-compact and $\mathcal{U}$ is a filter on X then*

$$\inf_{x\in A} \sup_{U\in\mathcal{U}} \delta(x,U) = \sup_{U\in\mathcal{U}} \inf_{x\in A} \delta(x,U).$$

Proof. One inequality is clear. To prove the other inequality suppose that $\sup_{U\in\mathcal{U}} \inf_{x\in A} \delta(x,U) < l$. Then it follows that for each $U \in \mathcal{U}$ we can find $a \in A$ such that $\delta(a,U) \leqslant l$, i.e. $U^{(l)} \cap A \neq \emptyset$. This implies that $\{U^{(l)} \cap A \mid U \in \mathcal{U}\}$ generates a filter on A. Since A is 0-compact it follows that, for any $\varepsilon > 0$, there exists $x \in A$ such that

$$\begin{aligned}\sup_{U\in\mathcal{U}} \delta(x,U) &\leqslant \sup_{U\in\mathcal{U}} \delta\big(x, U^{(l)}\big) + l\\ &\leqslant \sup_{U\in\mathcal{U}} \delta\big(x, U^{(l)} \cap A\big) + l\\ &\leqslant \varepsilon + l,\end{aligned}$$

which by the arbitrariness of ε proves the lemma. □

Proposition 3.3. *For approach spaces X and Y and $f : X \to Y$ the following are equivalent*:

(1) *f is a proper contraction,*
(2) *f is a closed contraction and for each $y \in Y$, $f^{-1}(y)$ is 0-compact,*
(3) *for each $\mathcal{U} \in \mathbf{U}(X)$: $f(\lambda\mathcal{U}) = \lambda f(\mathcal{U})$.*

Proof. That (1) implies (2) follows at once from Propositions 2.10(3) and 3.1 by considering the pullback diagram

$$\begin{array}{ccc} f^{-1}(y) & \xrightarrow{f|_{f^{-1}(y)}} & \{y\}\\ \downarrow & & \downarrow\\ X & \xrightarrow{f} & Y\end{array}$$

To prove that (2) implies (3) let $\mathcal{U} \in \mathbf{U}(X)$. Note that one inequality follows at once from the fact that f is a contraction. To show the other inequality let $y \in Y$ then, invoking Lemma 3.2

$$\begin{aligned}f(\lambda\mathcal{U})(y) &= \inf_{x\in f^{-1}(y)} \sup_{U\in\mathcal{U}} \delta(x,U)\\ &= \sup_{U\in\mathcal{U}} \inf_{x\in f^{-1}(y)} \delta(x,U)\\ &= \sup_{U\in\mathcal{U}} f(\delta_U)(y)\end{aligned}$$

$$\begin{aligned}&\leqslant \sup_{U\in\mathcal{U}} \delta_{f(U)}(y)\\ &= \lambda f(\mathcal{U})(y).\end{aligned}$$

To prove that (3) implies (1) note that f is clearly a contraction. To prove that it is a proper contraction let Z be an arbitrary approach space and consider the map $f \times 1_Z : X \times Z \to Y \times Z$. First, for any ultrafilter $\mathcal{U}$ on $X \times Z$ and $(y, z) \in Y \times Z$ we have

$$\begin{aligned}
f \times 1_Z(\lambda\mathcal{U})(y,z) &= \inf_{x\in f^{-1}(y)} \lambda\mathcal{U}(x,z)\\
&= \inf_{x\in f^{-1}(y)} \lambda \mathrm{pr}_1\mathcal{U}(x) \vee \lambda \mathrm{pr}_2\mathcal{U}(z)\\
&= f(\lambda \mathrm{pr}_1\mathcal{U})(y) \vee 1_Z(\lambda \mathrm{pr}_2\mathcal{U})(z)\\
&\leqslant \lambda(f\mathrm{pr}_1\mathcal{U})(y) \vee \lambda(1_Z \mathrm{pr}_2\mathcal{U})(z)\\
&= \lambda\big(\mathrm{pr}_1(f \times 1_Z\mathcal{U})\big)(y) \vee \lambda\big(\mathrm{pr}_2(f \times 1_Z\mathcal{U})\big)(z)\\
&= \lambda f \times 1_Z(\mathcal{U})(y,z).
\end{aligned}$$

Then, if $A \subset X \times Z$ and $(y, z) \in Y \times Z$ it follows that

$$\begin{aligned}
f \times 1_Z(\delta_A)(y,z) &= \inf_{x\in f^{-1}(y)} \delta\big((x,z),A\big)\\
&= \inf_{x\in f^{-1}(y)} \inf_{\substack{\mathcal{U}\text{ ultra}\\ A\in\mathcal{U}}} \lambda\mathcal{U}(x,z)\\
&= \inf_{\substack{\mathcal{U}\text{ ultra}\\ A\in\mathcal{U}}} f \times 1_Z(\lambda\mathcal{U})(y,z)\\
&\leqslant \inf_{\substack{\mathcal{U}\text{ ultra}\\ A\in\mathcal{U}}} \lambda(f \times 1_Z\mathcal{U})(y,z)\\
&\leqslant \inf_{\substack{\mathcal{W}\text{ ultra}\\ f\times 1_Z(A)\in\mathcal{W}}} \lambda\mathcal{W}(y,z)\\
&= \delta_{f\times 1_Z(A)}(y,z). \qquad \square
\end{aligned}$$

Proposition 3.4. *If $f : X \to Y$ is a surjective proper contraction then $\mu_c(X) = \mu_c(Y)$.*

Proof. One inequality follows from Proposition 1.4. The other inequality follows from

$$\begin{aligned}
\mu_c(X) &= \sup_{\mathcal{U}\in\mathbf{U}(X)} \inf_{x\in X} \lambda\mathcal{U}(x)\\
&= \sup_{\mathcal{U}\in\mathbf{U}(X)} \inf_{y\in Y} \inf_{x\in f^{-1}(y)} \lambda\mathcal{U}(x)\\
&= \sup_{\mathcal{U}\in\mathbf{U}(X)} \inf_{y\in Y} f(\lambda\mathcal{U})(y)\\
&= \sup_{\mathcal{U}\in\mathbf{U}(X)} \inf_{y\in Y} \lambda f(\mathcal{U})(y)\\
&\leqslant \sup_{\mathcal{W}\in\mathbf{U}(Y)} \inf_{y\in Y} \lambda\mathcal{W}(y) = \mu_c(Y). \qquad \square
\end{aligned}$$

Corollary 3.5. *If $f : X \to Y$ is a proper contraction then for every* 0*-compact subset $B \subset Y$ also $f^{-1}(B)$ is* 0*-compact.*

Proof. Let $f : X \to Y$ be a proper contraction and let $B \subset Y$ be 0-compact. Then in view of Proposition 2.10

$$f|_{f^{-1}(B)} : f^{-1}(B) \to B$$

is proper. Consider the proper contraction $\pi : B \to P$ where P is a one-point space. Then applying Propositions 3.3 and 2.10 also the composition $\pi \circ f|_{f^{-1}(B)}$ is proper. In view of Proposition 3.1 this implies that $f^{-1}(B)$ is 0-compact. □

Theorem 3.6. *An approach space X is* 0*-compact if and only if for any one-point space P the unique morphism $\pi : X \to P$ is a proper contraction and consequently the notion of* 0*-compactness coincides with $\mathcal{F}$-compactness in the sense of* [2].

Proof. Since $\pi : X \to P$ is always a closed contraction this follows at once from Proposition 3.3. □

Theorem 3.7 (Kuratowski–Mrówka). *An approach space X is* 0*-compact if and only if for any approach space Z the projection $\mathrm{pr}_Z : X \times Z \to Z$ is closed.*

Proof. Any pullback of $\pi : X \to P$, where P is a one-point space, is given by a diagram

$$\begin{array}{ccc} X \times Z & \xrightarrow{\mathrm{pr}_Z} & Z \\ {\scriptstyle \mathrm{pr}_X}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\pi} & P \end{array}$$

and hence the result follows at once. □

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